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Exercice ①:

- $P(z) = z^3 - (1+2\cos\theta) z^2 + (1+2\cos\theta) z - 1$.

1). $P(1) = 1^3 - (1+2\cos\theta)(1)^2 + (1+2\cos\theta)(1) - 1$

- $P(1) = 1 - 1 - 2\cos\theta + 1 + 2\cos\theta - 1$

- $P(1) = 0$.

- Soit $P(z) = (z-1)(az^2+bz+c)$.

	1	$-1-\cos\theta$	$1+2\cos\theta$	-1
1	$\downarrow$	1	$-2\cos\theta$	1
	1	$-2\cos\theta$	1	0

$\Longrightarrow P(z) = (z-1)(z^2 - 2\cos\theta z + 1)$

- $P(z) = 0 \Longrightarrow$

$z - 1 = 0$

$z = 1$

$\Longrightarrow \boxed{z_0 = 1}$

$z^2 - 2\cos\theta z + 1 = 0$

$\Delta = b^2 - 4ac$

$\Delta = (-2\cos\theta)^2 - 4 \times 1 \times 1$

$\Delta = 4\cos^2\theta - 4$

$\Delta = 4(\cos^2\theta - 1)$

$\Delta = -4\sin\theta$

$\Delta = 4i^2\sin^2\theta$

$\Longrightarrow \sqrt{\Delta} = 2i\sin\theta$.

$z = \dfrac{2\cos\theta - 2i\sin\theta}{2}$ et

$z = \dfrac{2\cos\theta + 2i\sin\theta}{2}$

$\Longrightarrow \boxed{z_1 = \cos\theta - i\sin\theta}$

et $\boxed{z_2 = \cos\theta + i\sin\theta}$

2) • $M_1(\cos\theta, \sin\theta)$

$x_{M_1}^2 + y_{M_2}^2 = \cos^2\theta + \sin^2\theta$

$= 1.$

$\Rightarrow$ M_1 décrit le cercle de centre $O(0,0)$ et de rayon 1.

• $M_2(\cos\theta, -\sin\theta)$

$x_{M_2}^2 + y_{M_2}^2 = \cos^2\theta + (-\sin\theta)^2$

$= 1$

$\Rightarrow$ M_2 décrit le cercle de centre $O(0,0)$ et de rayon 1.

3)

a) • $z_G = \dfrac{z_0 + z_1 - 3z_2}{(-1)}$

• $z_G = 3z_2 - z_1 - z_0$

$z_G = 3(\cos\theta - i\sin\theta)$

$- \cos\theta - i\sin\theta - 1.$

• $z_G = 2\cos\theta - 1 - 4i\sin\theta.$

$\Rightarrow \begin{cases} x_G = 2\cos\theta - 1 \\ y_G = -4\sin\theta. \end{cases}$

$\Rightarrow \begin{cases} \cos\theta = \dfrac{x+1}{2} \\ \sin\theta = \dfrac{-y}{4} \end{cases}$

$\Rightarrow \dfrac{(x+1)^2}{2^2} + \dfrac{y^2}{4^2} = 1$

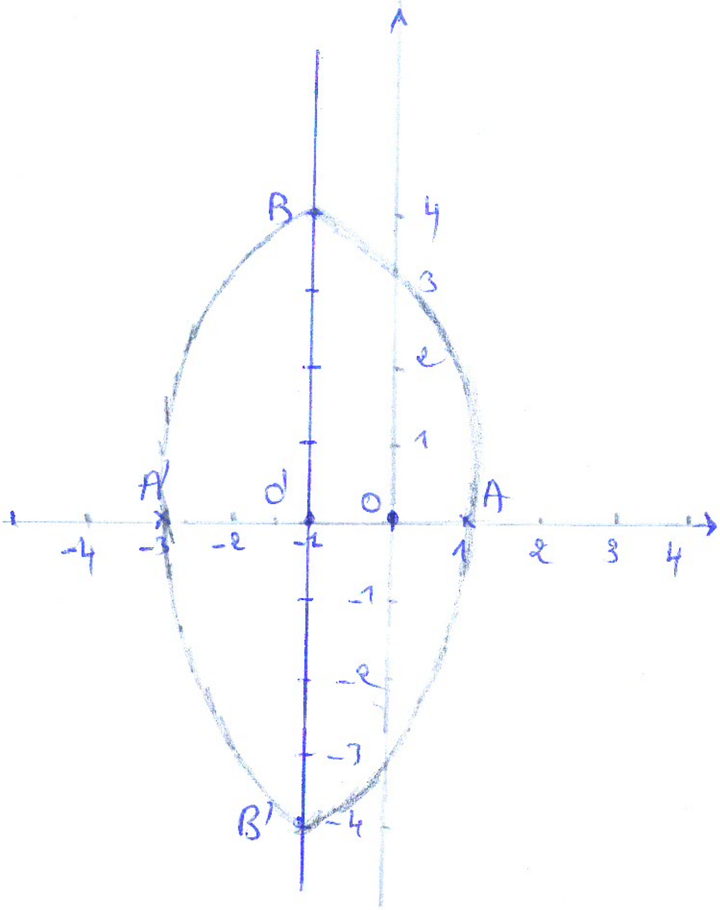
$\Rightarrow$ Γ est une ellipse.

b) de centre : $O'(-1,0)$.

des sommets : $A(2,0)$, $A'(-2,0)$, $B(0,4)$, $B'(0,-4)$.

$c = \sqrt{16-4} = \sqrt{12} = 2\sqrt{3}$

$e = \dfrac{c}{b} = \dfrac{2\sqrt{3}}{4} = \dfrac{\sqrt{3}}{2}.$



4) $\theta = \frac{\pi}{2}$.

a) $M_0(1,0)$

$z_{M_1} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$

- $M_1(0,1)$.

M_2 $z_2 = \cos\frac{\pi}{2} - i\sin\frac{\pi}{2}$

- $M_2(0,-1)$

- $z_G = 2\cos\frac{\pi}{2} - 1 - 4i\sin\frac{\pi}{2}$

- $G(-1,-4)$.

$G = B'$, G est un sommet de l'ellipse.

b) $MM_0^2 + MM_1 - 3MM_2^2 = 6$.

$\varphi(M) = MM_0^2 + MM_1 - 3MM_2^2$

$\varphi(G) = GM_0^2 + GM_1^2 - 3GM_2^2$

$\varphi(G) = |z_0 - z_G|^2 + |z_1 - z_G|^2 - 3|z_0 - z_G|^2$

$\Rightarrow \varphi(G) = 16$. or

$\cancel{4)}$ $G = \text{bar} \begin{array}{c|c|c} M_0 & M_1 & M_2 \\ \hline 1 & 1 & -3 \end{array}$

$\Rightarrow \varphi(G) - MG^2 = 6$

$\Rightarrow MG^2 = 10$

$\Rightarrow \Gamma'$ est le cercle de centre G et de rayon $\sqrt{10} = GM_2$.

Exercice ②:

$$\begin{cases} f(x) = x(1-\ln x) \\ f(0) = 0. \end{cases}$$

1) -

a) - • $\lim\limits_{x\to0^+} f(x) = \lim\limits_{x\to0^+} x(1-\ln x)$

$= \lim\limits_{x\to0^+} x - x\ln x = 0-0=0$

$\Rightarrow \lim\limits_{x\to0^+} f(x) = f(0)$

$\Rightarrow$ f est continue à droite de $x_0=0$.

• $\lim\limits_{x\to0^+} \frac{f(x)-0}{x-0} = \lim\limits_{x\to0^+} \frac{f(x)}{x}$

$= \lim\limits_{x\to0^+} \frac{\cancel{x}(1-\ln x)}{\cancel{x}}$

$= \lim\limits_{x\to0^+} 1-\ln x = 1-(-\infty)$

$= 1+\infty = +\infty$

$\Rightarrow \lim\limits_{x\to0^+} \frac{f(x)-0}{x-0} = +\infty \neq 0 \neq f(0)$

$\Rightarrow$ f n'est pas dérivable à droite de $x_0=0$

$\Rightarrow O(0,0)$ f admet une tangente verticale.

b). $f'(x) = (x(1-\ln x))'$

$f'(x) = 1 - \ln x - \frac{1}{\cancel{x}}\cancel{x}$

$f'(x) = -\ln x = 0 \Rightarrow \boxed{x=1}$.

• $\lim\limits_{x\to+\infty} f(x) = \lim\limits_{x\to+\infty} x(1-\ln x)$

$= +\infty \times -\infty = -\infty$

x	0		1		$+\infty$
$f'(x)$		$+$	0	$-$	
$f(x)$	0	$\nearrow$	1	$\searrow$	$-\infty$

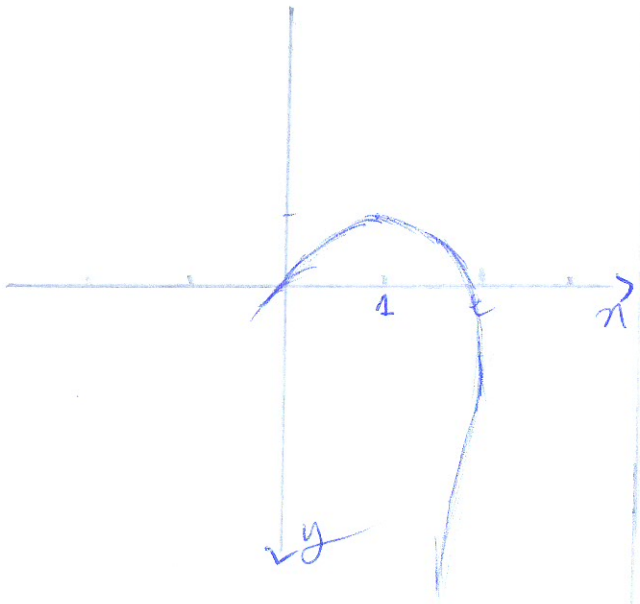
c) $\lim\limits_{x\to+\infty} \frac{f(x)}{x} = \lim\limits_{x\to+\infty} \frac{x(1-\ln x)}{x}$

$= \lim\limits_{x\to+\infty} 1-\ln x$

$= 1-\infty = -\infty.$

$\Rightarrow \lim\limits_{x\to+\infty} \frac{f(x)}{x} = -\infty$

$\Rightarrow$ C admet une bp de direction oy.



2) $f_n(x) = x^n(1-\ln x)$;

$f_n(0)=0$

a) $\lim\limits_{x\to 0} \frac{f_n(x)-f_n(0)}{x-0} =$

$\lim\limits_{x\to 0} \frac{x^n(1-\ln x)}{x} =$

$\lim\limits_{x\to 0} x^{n-1}(1-\ln x)$

$= \lim\limits_{x\to 0} x^{n-1} - x^{n-1}\ln x$

$= 0-0 = 0.$ <u>or</u>

$f_n(0)=0 \Rightarrow$

$f_n(x)$ est dérivable à droite de $\boxed{x_0=0}$

$\Rightarrow$ f_n admet une demi-tangente à droite de 0 horizontale d'équation : $\boxed{y=0}$

b) • $\lim\limits_{x\to 0} f_n(x) = \lim\limits_{x\to 0} x^n(1-\ln x)$

$\Rightarrow \lim\limits_{x\to 0} x^n - x^n\ln x$

$= 0-0 = 0.$

• $\lim\limits_{x \to +\infty} f(x) = \lim\limits_{x \to +\infty} x^n(1-\ln x)$

$= +\infty(1 - \infty) =$

$+\infty \times -\infty = -\infty$

• $(f_n(x))' = (x^n(1-\ln x))'$

$= (x^n - x^n \ln x)'$

$= n x^{n-1} - ((x^n)' \ln x + x^n (\ln x)')$

$= n x^{n-1} - n x^{n-1} \ln x - \frac{x^n}{x}$

$= n x^{n-1} - n x^{n-1} \ln x - x^{n-1}$

$= x^{n-1}(n - n \ln x - 1)$

$f_n'(x) = x^{n-1}(n(1-\ln x) - 1)$.

• $f_n'(x) = 0 \implies$

$x^{n-1} = 0$ ou $n(1-\ln x) - 1 = 0$

$\boxed{x=0}$ | $n(1-\ln x) = 1$

$-\ln x = \frac{1}{n} - 1$

$\boxed{x = e^{\frac{n-1}{n}}}$

$f(e^{\frac{n-1}{n}}) = \frac{e^{n-1}}{n}$.

x	0		$e^{\frac{n-1}{n}}$		$+\infty$
$f_n'(x)$		$+$	0	$-$	
$f_n(x)$	0	$\nearrow$	$\frac{e^{n-1}}{n}$	$\searrow$	$-\infty$

3) -

a)- $f_{n+1}(x) = f_n(x)$

$x^{n+1}(1-\ln x) = x^n(1-\ln x)$

$x^{n+1}(1-\ln x) - x^n(1-\ln x) = 0$

$x^n(x(1-\ln x) - (1-\ln x)) = 0$

$x^n(x-1)(1-\ln x) = 0$

$x^n = 0$, $x = 0$ | $x - 1 = 0$, $x = 1$ | $1 - \ln x = 0$, $\ln x = 1$, $x = e$

• $x = 0 \implies f_n(0) = 0 \implies (0,0)$.

• $x = 1 \implies f_n(1) = 1 \implies (1,1)$.

• $x = e \implies f_n(e) = 0 \implies (e,0)$.

b) - $f_{n+1}(x) - f_n(x) = 0$

$\implies x = 0 \mid x = 1 \mid x = e$.

x	0		1		e		$+\infty$
x^n		$+$		$+$		$+$	
$x-1$		$-$	0	$+$		$+$	
$1-\ln x$		$+$		$+$	0	$-$	
$f_{n+1}(x) - f_n(x)$		$-$		$+$		$-$	

4) • $U_n = \int_{1/e}^{1} f_n(x)\,dx$.

a) - U_n est la surface compris entre C_n, l'axe des abscisses et les droites d'equation $x=1$ et $x=\frac{1}{e}$

Comme C_n est au dessous de ox dans l'intervalle $[\frac{1}{e};1]$.

b) D'après le tableau de variation de f_n est positive sur $[\frac{1}{e},1]$

D'one U_n est positive

• D'après la position relative $f_{n+1}-f_n < 0$ sur $[\frac{1}{e},1]$

$\Rightarrow U_{n+1} - U_n < 0 \Rightarrow$ U_n est décroissante.

c). $U_n = \int_{1/e}^{1} f_n(x)\,dx$.

• $U_n = \int_{1/e}^{1} x^n(1-\ln x)\,dx$

on pose $\begin{cases} u(x) = 1-\ln x \\ v'(x) = x^n \end{cases}$

$\Rightarrow$

• $U_n = \left[(1-\ln x)\frac{1}{n+1}x^{n+1}\right]_{1/e}^{1} - \int_{1/e}^{1} -\frac{1}{x}\times\frac{1}{n+1}\times x^{n+1}\,dx$

$\Rightarrow$

• $U_n = \left[\frac{1-\ln x}{n+1}x^{n+1}\right]_{1/e}^{1} - \int_{1/e}^{1} \frac{-x\times x^n}{x\times n+1}\,dx$.

• $U_n = \left[\frac{1-\ln x}{n+1}x^{n+1}\right]_{1/e}^{1} + \frac{1}{n+1}\int_{1/e}^{1} x^n\,dx$

- $u_n = \frac{1}{n+1}\left[(1-\ln x)\, x^{n+1}\right]_{1/e}^{1}$

$+ \frac{1}{n+1}\left[\frac{1}{n+1} x^{n+1}\right]_{1/e}^{1}$

$\Longrightarrow$

- $u_n = \frac{1}{n+1}\left((1-\ln 1)(1)^{n+1} - (1-\ln 1/e)(1/e)^{n+1}\right)$

$+ \frac{1}{n+1}\left(\frac{1}{n+1}(1)^{n+1} - \frac{1}{n+1}\left(\frac{1}{e}\right)^{n+1}\right)$

- $u_n = \frac{1}{n+1}\left(1 - \frac{2}{e^{n+1}}\right)$

$+ \frac{1}{n+1}\left(\frac{1}{n+1} - \frac{1}{n+1} \times \frac{1}{e^{n+1}}\right)$

- $u_n = \frac{1}{n+1}\left(1 - \frac{2}{e^{n+1}}\right)$

$+ \frac{1}{(n+1)^2}\left(1 - \frac{1}{e^{n+1}}\right).$

- $\lim\limits_{n\to+\infty} u_n = \lim \frac{1}{n+1}\left(1 - \frac{2}{e^{n+1}}\right)$

$+ \frac{1}{(n+1)^2}\left(1 - \frac{1}{e^{n+1}}\right).$

- $\lim\limits_{n\to+\infty} \frac{1}{n+1} = 0$

- $\lim\limits_{n\to+\infty} 1 - \frac{2}{e^{n+1}} = 1 - 0 = 1$

- $\lim \frac{1}{(n+1)^2} = 0$

- $\lim\limits_{n\to+\infty} 1 - \frac{1}{e^{n+1}} = 1 - 0 = 1$

$\Longrightarrow \lim\limits_{n\to+\infty} u_n = 0 \times 1 + 0 \times 1$

$= 0$

$\Longrightarrow$

$$\boxed{\lim_{n\to+\infty} u_n = 0}$$

Exercice ③ :

1)- $D_f = \mathbb{R}$.

$$f(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

a)- • $\lim\limits_{x \to -\infty} f(x) = \lim\limits_{x \to -\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}}$

$$= \frac{e^{-\infty} - e^{+\infty}}{e^{-\infty} + e^{+\infty}} = \frac{0 - \infty}{0 + \infty}$$

$$= \frac{\infty}{\infty} = \text{F.I.}$$

• $\lim\limits_{x \to -\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} = \lim\limits_{x \to -\infty} \frac{e^x - 1/e^x}{e^x + 1/e^x}$

$$= \lim_{x \to -\infty} \frac{e^{2x} - \cancel{1/x}}{e^{2x} + \cancel{1/x}}$$

$$= \lim_{x \to -\infty} \frac{e^{2x} - 1}{e^{2x} + 1} = \frac{0 - 1}{0 + 1}$$

$$\lim_{x \to -\infty} f(x) = -1$$

$\Rightarrow y = -1$: A.H

• $\lim\limits_{x \to +\infty} f(x) = \lim\limits_{x \to +\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}}$

$$= \lim_{x \to +\infty} \frac{e^x(1 - e^{-2x})}{e^x(1 + e^{-2x})} = \frac{1 - 0}{1 + 0} = 1$$

• $\lim\limits_{x \to +\infty} f(x) = 1$

$\Rightarrow y = 1$: A.H

b) $f(-x) = \frac{e^{-x} - e^x}{e^{-x} + e^x}$

$\Rightarrow f(-x) = -\frac{(e^x - e^{-x})}{e^x + e^{-x}}$

$= -f(x)$. $\Rightarrow$

$f(-x) = -f(x)$

$\Rightarrow$ f est impaire.

$$f'(x) = \left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)' =$$

$$\frac{(e^x - e^{-x})'(e^x + e^{-x}) - (e^x - e^{-x})(e^x + e^{-x})'}{(e^x + e^{-x})^2}$$

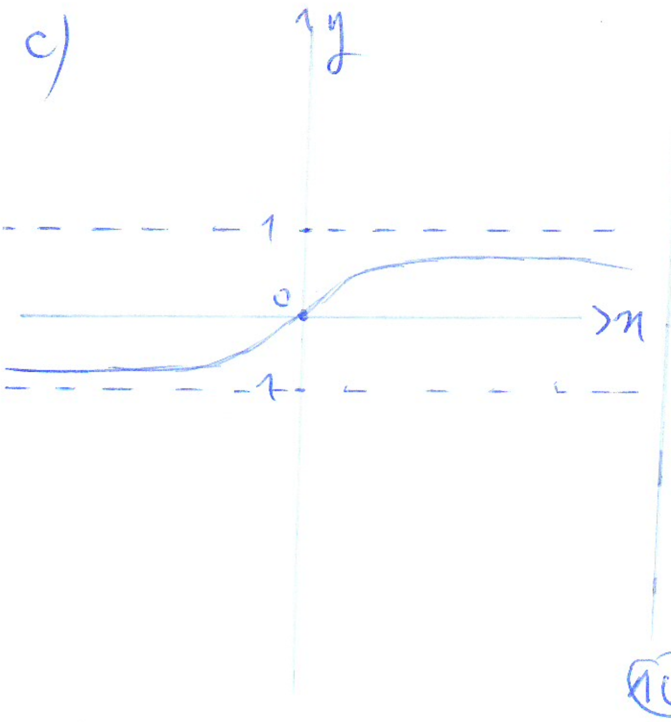
$$\Rightarrow \frac{(e^x+e^{-x})(e^x+e^{-x}) - (e^x-e^{-x})(e^x-e^{-x})}{(e^x+e^{-x})^2}$$

• $f'(x) = \dfrac{4}{(e^x+e^{-x})^2}$.

$\Rightarrow f'(x) > 0$.

x	$-\infty$		$+\infty$
$f'(x)$		$+$	
$f(x)$	-1	$\nearrow$	1

c)



d) $A = \int_0^{\ln 3} f(x)\,dx$.

$= \int_0^{\ln 3} \dfrac{e^x - e^{-x}}{e^x + e^{-x}}\,dx$.

$= \left[\ln(e^x + e^{-x})\right]_0^{\ln 3}$

$= \ln(e^{\ln 3} + e^{-\ln 3}) - \ln(e^0 + e^0)$

$= \ln(3 + \frac{1}{3}) - \ln(2)$.

$= \ln(\frac{10}{3}) - \ln 2$.

$\Rightarrow \boxed{A = \ln \frac{5}{3}}$.

2)-

a)- $u_1 = \int_0^{\ln 3} f(t)\,dt$

$\Rightarrow u_1 = A = \ln \frac{5}{3}$.

b)- Mq : $0 \le u_n \le (\frac{4}{5})^n \ln 3$

f est croissante sur $[0, \ln 3]$.

$\Longrightarrow$

- $f(0) < f(t) < f(\ln 3)$
- $0 < f(t) < \frac{4}{5}$
- $0 < f(t)^n < \left(\frac{4}{5}\right)^n$
- $0 < u_n < \int_0^{\ln 3} \left(\frac{4}{5}\right)^n$
- $0 < u_n < \left(\frac{4}{5}\right)^n \ln(3)$.
- $\lim\limits_{n \to +\infty} u_n = \lim\limits_{n \to +\infty} \left(\frac{4}{5}\right)^n = 0$

D'aprés gendorme.

c) Mq : $1 - f'(x) = f^2(x)$

$$f^2(x) = \frac{e^{2x} - 2e^{x}e^{-x} + e^{-2x}}{e^{2x} + 2e^{x}e^{-x} + e^{-2x}}$$

$$= \frac{e^{2x} + e^{-2x} - 2}{e^{2x} + e^{-2x} + 2}$$

$$1 - f'(x) = 1 - \frac{4}{e^{2x} + e^{-2x} + 2}$$

$$= \frac{e^{2x} + e^{-2x} - 2}{e^{2x} + e^{-2x} + 2} = f^2(x)$$

$$\Longrightarrow \boxed{1 - f'(x) = f^2(x)}.$$

- Mq : $u_{n+2} - u_n = \frac{-1}{n+1}\left(\frac{4}{5}\right)^{n+1}$

$u_{n+2} - u_n =$

$$\int_0^{\ln 3} f(t)^{n+2} - f(t)^n \, dt$$

$$= \int_0^{\ln 3} f(t)^n \left(f^2(t) - 1\right) dt$$

$$= - \int_0^{\ln 3} f(t)^n f'(t) \, dt$$

$$= - \left[\frac{1}{n+1} f(t)^{n+1}\right]_0^{\ln 3}$$

$$= \frac{-1}{n+1} f(\ln(3))^{n+1}$$

$$= \frac{-1}{n+1} \left(\frac{4}{5}\right)^{n+1}.$$

$$\Longrightarrow$$

$$\boxed{u_{n+2} - u_n = \frac{-1}{n+1}\left(\frac{4}{5}\right)^{n+1}}$$

d/- • $U_k - U_{k-2} = \frac{-1}{k-1}\left(\frac{4}{5}\right)^{k-1}$

avec $k = 2p$.

• $p=1$: $\cancel{U_2} - U_0 = \frac{-1}{2\times1-1}\left(\frac{4}{5}\right)^{2\times1-1}$

• $p=2$: $U_4 - \cancel{U_2} = \frac{-1}{2\times2-1}\left(\frac{4}{5}\right)^{2\times2-1}$

$\vdots$

• $p=2n$: $U_{2n} - \cancel{U_{2n-2}} = \frac{-1}{2n-1}\left(\frac{4}{5}\right)^{2n-1}$

$\Longrightarrow$

$$U_{2n} - U_0 = \sum_{p=1}^{n} \frac{-1}{2p-1}\left(\frac{4}{5}\right)^{2p-1}$$

or $U_0 = \int_0^{\ln 3} (f(t))^0 dt$

$= [t]_0^{\ln 3} = \ln 3$

$$\Longrightarrow U_{2n} = \ln 3 - \sum_{p=1}^{n} \frac{1}{2p-1}\left(\frac{4}{5}\right)^{2p-1}$$

• $U_k - U_{k-2} = \frac{-1}{k-1}\left(\frac{4}{5}\right)^{k-1}$

avec $k = 2p+1$

$\Longrightarrow U_{2p+1} - U_{2p-1} = \frac{-1}{2p}\left(\frac{4}{5}\right)^{2p}$

• $p=1$: $U_3 - U_1 = \frac{-1}{2\times1}\left(\frac{4}{5}\right)^{2\times1}$

• $p=2$: $U_5 - U_3 = \frac{-1}{2\times2}\left(\frac{4}{5}\right)^{2\times2}$

$\vdots$

$p=2n$: $U_{2n+1} - U_{2n-1} = \frac{-1}{2n}\left(\frac{4}{5}\right)^{2n}$

$$\Longrightarrow U_{2n+1} - U_1 = \sum_{p=1}^{n} \frac{-1}{2p}\left(\frac{4}{5}\right)^{2p}$$

or $U_1 = \ln\frac{5}{3}$.

$$\Longrightarrow U_{2n+1} = \ln\frac{5}{3} - \sum_{p=1}^{n} \frac{1}{2p}\left(\frac{4}{5}\right)^{2p}$$

e/- $S_n = \sum_{p=1}^{2n} \frac{1}{p}\left(\frac{4}{5}\right)^p$

$= \sum_{p=1}^{n} \frac{1}{2p}\left(\frac{4}{5}\right)^{2p} +$

$\sum_{p=1}^{n} \frac{1}{2p-1}\left(\frac{4}{5}\right)^{2p-1}$

Or : $\sum_{p=1}^{n} \frac{1}{2p-1}\left(\frac{4}{5}\right)^{2p-1} = -U_{2n} + \ln 3$

et $\sum_{p=1}^{n} \frac{1}{2p}\left(\frac{4}{5}\right)^{2p} = -U_{2n+1} + \ln\frac{5}{3}$

$\Longrightarrow S_n = -U_{2n} + \ln 3 - U_{2n+1} + \ln 5 - \ln 3$

• $\lim_{n\to+\infty} U_n = 0 \Longrightarrow$

$$\boxed{\lim_{n\to+\infty} S_n = \ln 5}$$