Nombres Complexes:

Mejeich/Dahnein
Mouhaye Mialo

Exercice 10

Soit $z = e^{i\frac{2\pi}{7}}$. On pose $\alpha = z + z^2 + z^4$.

1.Calculer $\alpha + \overline{\alpha}$ et $\alpha\overline{\alpha}$.

2. En déduire que : $\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{8\pi}{7} = -\frac{1}{2}$;

et que $\sin\frac{2\pi}{7} + \sin\frac{4\pi}{7} + \sin\frac{8\pi}{7} = \frac{\sqrt{7}}{2}$.

$Z = e^{i\frac{2\pi}{7}}$

$\alpha = Z + Z^2 + Z^4$

$\overline{\alpha} = \overline{Z} + \overline{Z}^2 + \overline{Z}^4$

1/ On remarque que $Z^7 = 1$

car $Z^7 = (e^{i\frac{2\pi}{7}})^7 = e^{i2\pi}$

On a :

$\overline{Z} = \frac{1}{Z} = \frac{Z^7}{Z} = Z^6$

$\overline{Z}^2 = \frac{1}{Z^2} = \frac{Z^7}{Z^2} = Z^5$

$\overline{Z}^4 = \frac{1}{Z^4} = \frac{Z^7}{Z^4} = Z^3$

Alors : $\overline{\alpha} = Z^6 + Z^5 + Z^3$

$\alpha + \overline{\alpha} = Z + Z^2 + Z^4 + Z^6 + Z^5 + Z^3$

$1 + \alpha + \overline{\alpha} = 1 + Z + Z^2 + Z^4 + Z^6 + Z^5 + Z^3$

$1 + \alpha + \overline{\alpha} = 1 + Z + Z^2 + Z^3 + Z^4 + Z^5 + Z^6$

$1 + \alpha + \overline{\alpha} = \frac{1 - Z^7}{1 - Z} \Rightarrow \alpha + \overline{\alpha} + 1 = 0$

D'où $\alpha + \overline{\alpha} = -1$

$\alpha\overline{\alpha} = (Z + Z^2 + Z^4)(Z^6 + Z^5 + Z^3)$

$\alpha\overline{\alpha} = Z^7 + Z^6 + Z^4 + Z^8 + Z^7 + Z^5 + Z^{10} + Z^9 + Z^7$

$Z^8 = Z.Z^7 = Z$; $Z^9 = Z^2 Z^7 = Z^2$

et $Z^{10} = Z^3.Z^7 = Z^3$

Donc $\alpha\overline{\alpha} = 2 + 1 + Z + Z^2 + Z^3 + Z^4 + Z^5 + Z^6$

$\alpha\overline{\alpha} = 2$

2/ On a : $\alpha = e^{i\frac{2\pi}{7}} + (e^{i\frac{2\pi}{7}})^2 + (e^{i\frac{2\pi}{7}})^4$

$\alpha = e^{i\frac{2\pi}{7}} + e^{i\frac{4\pi}{7}} + e^{i\frac{8\pi}{7}}$

$\alpha = (\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{8\pi}{7}) + i(\sin\frac{2\pi}{7} + \sin\frac{4\pi}{7} + \sin\frac{8\pi}{7})$

$Re(\alpha) = \frac{\alpha + \overline{\alpha}}{2} \Rightarrow Re(\alpha) = \frac{-1}{2}$

D'où $\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{8\pi}{7} = -\frac{1}{2}$

Alors $\alpha = -\frac{1}{2} + iy$ avec $y = \sin\frac{2\pi}{7} + \sin\frac{4\pi}{7} + \sin\frac{8\pi}{7}$

On a : $\alpha\overline{\alpha} = 2 \Rightarrow |\alpha|^2 = 2 \Rightarrow (-\frac{1}{2})^2 + y^2 = 2$

$\Rightarrow \frac{1}{4} + y^2 = 2 \Rightarrow y^2 = 2 - \frac{1}{4} \Rightarrow y^2 = \frac{7}{4}$

$\Rightarrow y = \pm\sqrt{\frac{7}{4}}$

On $\sin(\pi + \alpha) = -\sin\alpha$

Alors $\sin\frac{8\pi}{7} = -\sin\frac{\pi}{7}$ et

$y = \sin\frac{2\pi}{7} + \sin\frac{4\pi}{7} - \sin\frac{\pi}{7}$

La fonction sinus est croissante et positive sur $[0; \frac{\pi}{2}]$ alors $\sin\frac{2\pi}{7} > \sin\frac{\pi}{7}$

$\Rightarrow \sin\frac{2\pi}{7} - \sin\frac{\pi}{7} > 0$

Comme $\sin\frac{4\pi}{7} > 0$

On a alors $y > 0$ donc $y = \sqrt{\frac{7}{4}} = \frac{\sqrt{7}}{2}$

D'où $\sin\frac{2\pi}{7} + \sin\frac{4\pi}{7} + \sin\frac{8\pi}{7} = \frac{\sqrt{7}}{2}$